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# Finite-dimensional duality on the generalized Lie algebra $\mathfrak{sl}(2)_q$

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**Abstract**
We show a local duality between the left and right finite-dimensional representations on the enveloping algebra of $\mathfrak{sl}(2)_q$, the generalized Lie algebra introduced by Lyubashenko and Sudbery. This duality works in a general context of algebras satisfying certain algebraic properties; our goal in this paper is to prove that the enveloping algebra of $\mathfrak{sl}(2)_q$ satisfies these abstract properties.

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## 1. Introduction

In earlier papers (see [7, 10, 11]) we studied a local duality in a non-commutative framework which extends well known local dualities on commutative Gorenstein rings and enveloping algebras of finite-dimensional solvable Lie algebras (see [2]). This local duality is defined by the last point $E_\mu$ in the minimal injective resolution of $R$. In fact, we have identified this injective module with the underlying left module $R^0$, i.e. the finite dual coalgebra of $R$. As $R^0$ has the natural structure of a bimodule, we have proved that $E_\mu$ has also a right module structure in such a way that $E_\mu$ and $R^0$ are isomorphic bimodules. In the case of commutative Gorenstein rings there is a duality, given by the Ext functor; this duality was extended to a non-commutative framework of algebras satisfying certain special conditions, and it was shown that $E_\mu$, or equivalently $R^0$, represents this duality. I.e., the duality is defined by the functor $\mathrm{Hom}_R(-, R^0)$. As a consequence the categories of left and right finite-dimensional representations are dual. Therefore the left and right behaviour of these algebras is symmetric.

We showed in [10] when this duality, the *Bernstein duality*, can be represented by a bimodule and have characterized when this happens. Specifically, if $K$ is a field of characteristic zero and $R$ a $K$-algebra which satisfies the Auslander–Gorenstein and Cohen–Macaulay conditions, and $\mathrm{idim}(R) = \mu = \mathrm{GKdim}(R)$, then the functor $\mathrm{Ext}^\mu_R(-, R)$ defines a duality between the categories of left and right finite-dimensional $R$-modules. If, in addition, $R$ has the strong second-layer condition and the vector space dimensions of $R/P$ and $\mathrm{Ext}^{\mu}_R(R/P, R)$ coincide for every cofinite prime ideal $P$ of $R$, then this duality is defined by $R^0$.

There appear many examples of algebras satisfying all these properties in the previous works. For instance, we prove it in [11] for the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{sl}(2))$, $q$ being a root of unity. Now, we consider another kind of quantization of the Lie algebra $\mathfrak{sl}(2)$, the generalized Lie algebra $\mathfrak{sl}(2)_q$, as Lyubashenko and Sudbery define in [18]. Both authors start with the definition of the generalized Lie algebra. This is a vector space with a bilinear bracket operation and an auxiliary structure: *the generalized antisymmetrizer*, satisfying a version of anticommutativity and Jacobi identity. Once the generalized Lie algebra is defined, they construct its universal enveloping algebra. After this, they consider the quantized enveloping algebra $\mathcal{U}_q(\mathfrak{g})$, for any simple Lie algebra $\mathfrak{g}$, and prove that, in the case of $\mathfrak{g}$ being of type $A_n$, $\mathcal{U}_q(\mathfrak{sl}(n))$ contains a generalized Lie algebra of dimension $n^2 - 1$. It appears as a finite-dimensional subspace which is invariant under the adjoint action associated with the Hopf algebra structure of the quantized enveloping algebra. Its quantum bracket is constructed from this adjoint action. Specifically, they find this quantum Lie algebra inside the locally finite part of $\mathcal{U}_q(\mathfrak{sl}(n))$.

Lyubashenko and Sudbery describe explicitly the generalized Lie algebra $\mathfrak{sl}(2)_q$ and its enveloping algebra, $\mathcal{A} = \mathcal{U}(\mathfrak{sl}(2)_q)$, which can be obtained as a quotient of a related algebra $\mathcal{B}$. The $\mathbb{C}$-algebra $\mathcal{B}$ is generated by four generators, $X_0$, $X_+$, $X_-$ and $C$, with relations

$$\begin{aligned}
&q^2 X_0 X_+ - X_+ X_0 = qCX_+ \\
&q^{-2} X_0 X_- - X_- X_0 = -q^{-1} CX_- \\
&X_+ X_- - X_- X_+ = (q + q^{-1})(C - \lambda X_0) X_0 \\
&CX_\pm - X_\pm C = CX_0 - X_0 C = 0
\end{aligned}$$

where $\lambda = q - q^{-1}$. The algebra $\mathcal{A}$ is obtained from $\mathcal{B}$ adding the relation $C = 1$ (see [18]). We shall define $q$-numbers $[p]$ by $[p] := (q^p - q^{-p})/(q - q^{-1})$.

With this description and the construction of the finite-dimensional irreducible representations given by Dobrev and Sudbery [6] and Arnaudon [1], we are going to prove that the algebras $\mathcal{B}$ and $\mathcal{A}$ verify all the algebraic properties mentioned at the beginning of this section. As a consequence, the Bernstein duality will be satisfied and it will be represented by the underlying bimodule of the dual coalgebra in both algebras.

The paper is organized as follows. In the second section, we describe explicitly the centre of $\mathcal{B}$ and $\mathcal{A}$, showing that both algebras are finitely generated as modules over their centres. In the third section we verify in our algebras the Auslander-regular, Cohen–Macaulay and strong second-layer conditions and the equality of the injective and Gelfand–Kirillov dimensions. For this, we introduce an auxiliary algebra $S$ and describe all the three algebras as iterated Ore extensions using certain special bases. To complete the representation of the Bernstein duality by $R^0$, we prove in the fourth section the equality of the $\mathbb{C}$-dimension of $R/P$ and $\mathrm{Ext}^{\mu}_R(R/P, R)$, for any cofinite prime ideal $P$ of $R = \mathcal{B}$ and $\mathcal{A}$. To do this, we first prove the corresponding equality for every finite-dimensional simple $R$-module $M$, and after this we obtain the result for any quotient $R/P$ because of the bijection between finite-dimensional simple $R$-modules and cofinite prime ideals of $R$. We shall use some techniques of non-commutative Groebner bases in the computation of the $\mathbb{C}$-dimension of $\mathrm{Ext}^{\mu}_R(M, R)$.

## 2. The centre

Let us consider the following element of $\mathcal{B}$:

$$C_2' = X_0^2 + \frac{q}{q + q^{-1}} X_- X_+ + \frac{q^{-1}}{q + q^{-1}} X_+ X_-.$$

We have from [18, lemma 3.2] that

$$C^k X_-^m X_0^n X_+^p$$

with $k, m, n, p \in \mathbb{N}$ a linear basis of $\mathcal{B}$. Using the formulae

$$X_-X_+ = C_2' - q^{-1}CX_0 - q^{-2}X_0^2 = \lambda^{-2}\{-(C^2 - \lambda^2 C_2') + (1+q^{-2})C(C-\lambda X_0) - q^{-2}(C-\lambda X_0)^2\}, \tag{1}$$

$$X_+X_- = C_2' + qCX_0 - q^2X_0^2 = \lambda^{-2}\{-(C^2 - \lambda^2 C_2') + (1+q^2)C(C-\lambda X_0) - q^2(C-\lambda X_0)^2\}, \tag{2}$$

we may express all the common powers of $X_-$ and $X_+$ in each monomial in terms involving $X_0$, $C$ and $C_2'$ only. As a consequence, it can be proved that

$$X_-^{a_-} X_+^{a_+} X_0^{a_0} C^{b_1} C_2'^{b_2} \qquad \text{with} \quad a_\pm, a_0, b_1, b_2 \in \mathbb{N}, \; a_+a_- = 0, \tag{3}$$

is also a basis of $\mathcal{B}$ (see [1]). From these expressions it is easy to check that, as well as $C$, the element $C_2'$ is another central element of $\mathcal{B}$.

For the computation of the centre of $\mathcal{B}$, it is handiest to use another basis obtained from (3). Since

$$X_0 = -\frac{1}{\lambda}(C - \lambda X_0) + \frac{1}{\lambda}C$$

we have that

$$X_-^{a_-} X_+^{a_+} (C-\lambda X_0)^{a_0} C^{b_1} C_2'^{b_2} \qquad \text{with} \quad a_\pm, a_0, b_1, b_2 \in \mathbb{N}, a_+a_- = 0, \tag{4}$$

is a new system of $\mathbb{C}$-generators of $\mathcal{B}$. In fact, we may prove the linear independence of these elements, so (4) determines a new basis of $\mathcal{B}$ as a vector space over $\mathbb{C}$.

In the rest of the paper let us consider $q$ a root of unity. More precisely let $l$ be the smallest positive integer such that $q^{2l} = 1$ (we also require that $l > 1$).

In accordance with [1], the centre of $\mathcal{B}$ is given by

$$\mathbb{C}[C, C_2', X_+^l, (C-\lambda X_0)^l] + \mathbb{C}[C, C_2', X_-^l, (C-\lambda X_0)^l].$$

Hence $\mathcal{B}$ is a finitely generated module over its centre. To obtain this description of the centre we only need to impose that the elements in (4) commute with each one of the generators $X_+$, $X_-$, $(C - \lambda X_0)$, $C$ and $C_2'$ and to make use of the formulae

$$X_+X_-^k = X_-^k X_+ + \lambda^{-1}[k]q^{k-2}X_-^{k-1}(C-\lambda X_0)((1+q^2)C - (1+q^{2k})(C-\lambda X_0)) \tag{5}$$

(which can be obtained from equation (3.4$b$) in [6]),

$$(C-\lambda X_0)^k X_+ = q^{-2k}X_+(C-\lambda X_0)^k \qquad (C-\lambda X_0)X_+^k = q^{-2k}X_+^k(C-\lambda X_0) \tag{6}$$

and

$$X_-X_+^k = X_+^kX_- - \lambda^{-1}[k]q^{-k+2}X_+^{k-1}(C-\lambda X_0)((1+q^{-2})C - (1+q^{-2k})(C-\lambda X_0));$$
$$(C-\lambda X_0)^k X_- = q^{2k}X_-(C-\lambda X_0)^k;$$
$$(C-\lambda X_0)X_-^k = q^{2k}X_-^k(C-\lambda X_0),$$

obtained by applying to (5) and (6) the automorphism of $\mathbb{C}$-algebras $\varphi$ in $\mathcal{B}$

$$\varphi(X_+) = X_-,$$
$$\varphi(X_-) = X_+,$$
$$\varphi(X_0) = X_0,$$
$$\varphi(C) = C,$$
$$\varphi(C_2') = C_2',$$
$$\varphi(q) = -q^{-1}.$$

A simple recursion gives us

$$X_-^p X_+^p = \lambda^{-2p} \prod_{r=0}^{p-1} q^{-2r-1}\{-qD^2q^{2r} + (q+q^{-1})C(C-\lambda X_0) - q^{-1}(C-\lambda X_0)^2 q^{-2r}\} \qquad (7)$$

for every positive integer $p$, where $D$ is defined by $D^2 = C^2 - \lambda^2 C_2'$. Taking $p = l$ in (7) we obtain a relation between $C$, $C_2'$, $X_+^l$, $X_-^l$ and $(C - \lambda X_0)^l$, the generators of the centre of $\mathcal{B}$ (see [1]).

The algebra $\mathcal{A}$ is also a finitely generated module over its centre. We may check that the system of $\mathbb{C}$-generators obtained for $\mathcal{A}$ taking $C = 1$ in (4) is also linearly independent. Therefore

$$\{X_-^{a_-} X_+^{a_+} (1-\lambda X_0)^{a_0} C_2'^{b} : a_\pm, a_0, b \in \mathbb{N}, a_+a_- = 0\}$$

is a $\mathbb{C}$-basis for $\mathcal{A}$. Computations similar to those performed in $\mathcal{B}$ show us that the centre of $\mathcal{B}$ is given by

$$\mathbb{C}[C_2', X_+^l, (1-\lambda X_0)^l] + \mathbb{C}[C_2', X_-^l, (1-\lambda X_0)^l].$$

As a consequence, $\mathcal{A}$ is finitely generated as a module over its centre.

## 3. Homological properties and regularity conditions

We now study homological properties and regularity conditions of $\mathcal{B}$ and $\mathcal{A}$. We shall prove that these algebras are Auslander regular and Cohen–Macaulay and satisfy the strong second-layer condition and we shall also compute their global, injective and Gelfand–Kirillov dimensions. We introduce an auxiliary algebra $S$, from which $\mathcal{B}$ can be obtained as a quotient.

### *3.1. The algebra S*

Let us define $S$ as the $\mathbb{C}$-algebra with generators $X_1$, $X_2$, $X_3$, $X_4$ and $X_5$ and relations

$$\begin{array}{lll} X_2X_1 = X_1X_2 - \lambda^{-1}(q+q^{-1})X_3(X_4 - X_3) & & \\ X_3X_1 = q^{-2}X_1X_3, & X_4X_1 = X_1X_4, & X_5X_1 = X_1X_5 \\ X_3X_2 = q^2X_2X_3, & X_4X_2 = X_2X_4, & X_5X_2 = X_2X_5 \\ X_4X_3 = X_3X_4, & X_5X_3 = X_3X_5, & X_5X_4 = X_4X_5. \end{array}$$

For our purposes it is interesting to describe $S$ as an iterated Ore extension (see, for instance [19], for topics related to Ore extensions). Indeed

$$S = \mathbb{C}[X_3, X_4, X_5][X_2; \sigma_0][X_1; \sigma_1, \delta]$$

where

$$\sigma_0 : \mathbb{C}[X_3, X_4, X_5] \longrightarrow \mathbb{C}[X_3, X_4, X_5]$$
$$\sigma_0(X_3) = q^2X_3, \qquad \sigma_0(X_4) = X_4, \qquad \sigma_0(X_5) = X_5,$$

is an automorphism,

$$\sigma_1 : \mathbb{C}[X_3, X_4, X_5][X_2; \sigma_0] \longrightarrow \mathbb{C}[X_3, X_4, X_5][X_2; \sigma_0]$$
$$\sigma_1(X_2) = X_2, \qquad \sigma_1(X_3) = q^{-2}X_3, \qquad \sigma_1(X_4) = X_4, \qquad \sigma_1(X_5) = X_5,$$

is an automorphism, and

$$\delta : \mathbb{C}[X_3, X_4, X_5][X_2; \sigma_0] \longrightarrow \mathbb{C}[X_3, X_4, X_5][X_2; \sigma_0]$$
$$\delta(X_2) = -\lambda^{-1}(q+q^{-1})X_3(X_4 - X_3), \qquad \delta(X_3) = \delta(X_4) = \delta(X_5) = 0$$

is a $\sigma_1$-derivation. As a consequence, $S$ is a noetherian domain and a $\mathbb{C}$-basis of $S$ is given by $\{X_1^{\alpha_1}X_2^{\alpha_2}X_3^{\alpha_3}X_4^{\alpha_4}X_5^{\alpha_5} : \alpha_i \in \mathbb{N}\}$.

Let us now see that $\mathcal{B}$ can be obtained as a quotient of $S$. Identifying

$$X_1 \equiv X_+, \qquad X_2 \equiv X_-, \qquad X_3 \equiv C - \lambda X_0,$$
$$X_4 \equiv C, \qquad X_5 \equiv C_2',$$

we observe that all the relations of $S$ are also satisfied in $\mathcal{B}$. Another extra relation in $\mathcal{B}$ arises from formulae (1) and (2):

$$\begin{aligned} C_2' &= X_-X_+ + \lambda^{-2}\{C^2 - (1+q^{-2})C(C-\lambda X_0) + q^{-2}(C-\lambda X_0)^2\} \\ &= X_+X_- + \lambda^{-2}\{C^2 - (1+q^{2})C(C-\lambda X_0) + q^{2}(C-\lambda X_0)^2\}. \end{aligned}$$

Let us consider the element $p$ of $S$ given by

$$\begin{aligned} p &= X_5 - X_2X_1 - \lambda^{-2}(X_4^2 - (1+q^{-2})X_4X_3 + q^{-2}X_3^2) \\ &= X_5 - X_1X_2 - \lambda^{-2}(X_4^2 - (1+q^{2})X_4X_3 + q^{2}X_3^2). \end{aligned}$$

Then we may check that there exists an isomorphism of algebras

$$\frac{S}{Sp} \cong \mathcal{B}.$$

*3.2. Global and injective dimensions*

Algebras $\mathcal{B}$ and $\mathcal{A}$ may be also described as iterated Ore extensions. From the $\mathbb{C}$-basis $\{C^{\alpha_1}X_-^{\alpha_2}X_0^{\alpha_3}X_+^{\alpha_4} : \alpha_i \in \mathbb{N}\}$ of $\mathcal{B}$ obtained in [18, lemma 3.2], and using the equality $X_0 = -(1/\lambda)(C - \lambda X_0) + (1/\lambda)C$, we have that

$$\{C^{\alpha_1}(C-\lambda X_0)^{\alpha_2}X_-^{\alpha_3}X_+^{\alpha_4} : \alpha_i \in \mathbb{N}\}$$

is also a $\mathbb{C}$-basis of $\mathcal{B}$. Similarly as done for $S$, it can be checked that $\mathcal{B}$ may be described as an iterated Ore extension

$$\mathcal{B} = \mathbb{C}[C - \lambda X_0, C][X_-; \sigma_0][X_+; \sigma_1, \delta]$$

where

$$\sigma_0 : \mathbb{C}[C-\lambda X_0, C] \longrightarrow \mathbb{C}[C-\lambda X_0, C]$$
$$\sigma_0(C-\lambda X_0) = q^2(C-\lambda X_0), \qquad \sigma_0(C) = C$$

is an automorphism,

$$\sigma_1 : \mathbb{C}[C-\lambda X_0, C][X_-;\sigma_0] \longrightarrow \mathbb{C}[C-\lambda X_0, C][X_-;\sigma_0]$$
$$\sigma_1(X_-) = X_-, \qquad \sigma_1(C-\lambda X_0) = q^{-2}(C-\lambda X_0), \qquad \sigma_1(C) = C$$

is an automorphism, and

$$\delta : \mathbb{C}[C-\lambda X_0, C][X_-;\sigma_0] \longrightarrow \mathbb{C}[C-\lambda X_0, C][X_-;\sigma_0]$$
$$\delta(X_-) = -\lambda^{-1}(q+q^{-1})(C-\lambda X_0)(C-(C-\lambda X_0)),$$
$$\delta(C-\lambda X_0) = \delta(C) = 0$$

is a $\sigma_1$-derivation. In particular, $\mathcal{B}$ is a noetherian domain.

Analogously, since the algebra $\mathcal{A}$ may be obtained from $\mathcal{B}$ adding the relation $C = 1$, we have that $\{(1-\lambda X_0)^{\alpha_1}X_-^{\alpha_2}X_+^{\alpha_3} : \alpha_i \in \mathbb{N}\}$ is a $\mathbb{C}$-basis of $\mathcal{A}$, and $\mathcal{A}$ can be described as an iterated Ore extension

$$\mathcal{A} = \mathbb{C}[1-\lambda X_0][X_-;\sigma_0][X_+;\sigma_1,\delta]$$

where

$$\sigma_0 : \mathbb{C}[1-\lambda X_0] \longrightarrow \mathbb{C}[1-\lambda X_0]$$
$$\sigma_0(1-\lambda X_0) = q^2(1-\lambda X_0)$$

is an automorphism,

$$\sigma_1 : \mathbb{C}[1-\lambda X_0][X_-;\sigma_0] \longrightarrow \mathbb{C}[1-\lambda X_0][X_-;\sigma_0]$$
$$\sigma_1(X_-) = X_-, \qquad \sigma_1(1-\lambda X_0) = q^{-2}(1-\lambda X_0)$$

is an automorphism, and

$$\delta : \mathbb{C}[1-\lambda X_0][X_-;\sigma_0] \longrightarrow \mathbb{C}[1-\lambda X_0][X_-;\sigma_0]$$
$$\delta(X_-) = -\lambda^{-1}(q+q^{-1})(1-\lambda X_0)(1-(1-\lambda X_0)), \qquad \delta(1-\lambda X_0) = 0$$

is a $\sigma_1$-derivation, In particular, $\mathcal{A}$ is a noetherian domain.

The former description of $S$ (respectively $\mathcal{B}$; $\mathcal{A}$) and the existence of modules $M$ over $S$ (over $\mathcal{B}$; over $\mathcal{A}$) such that $\mathrm{Ext}^5_S(M,S) \neq 0$ ($\mathrm{Ext}^4_{\mathcal{B}}(M,\mathcal{B}) \neq 0$; $\mathrm{Ext}^3_{\mathcal{A}}(M,\mathcal{A}) \neq 0$) *as we shall see in the next section*, allows us, applying [19, theorem 7.5.3], to compute the injective and global dimensions of these algebras:

**Proposition 1.**

(1) $\mathrm{gldim}(S) = \mathrm{idim}(S) = 5$;
(2) $\mathrm{gldim}(\mathcal{B}) = \mathrm{idim}(\mathcal{B}) = 4$;
(3) $\mathrm{gldim}(\mathcal{A}) = \mathrm{idim}(\mathcal{A}) = 3$.

We refer to [19] for topics related to both homological dimensions.

### *3.3. Gelfand–Kirillov dimension*

In this subsection, we are going to compute the Gelfand–Kirillov dimension of the algebras $S$, $\mathcal{B}$ and $\mathcal{A}$. We refer to [19] for topics related to this dimension (see also [13]).

We start with $S$. This is an affine $\mathbb{C}$-algebra with generating subspace $V = \mathbb{C}\{X_1, X_2, X_3, X_4, X_5\}$. We consider the standard finite-dimensional filtration $\{S_i\}_i$ of $S$, with $S_0 = V^0 = \mathbb{C}$ and $S_i = \sum_{j=0}^{i} V^j$ for any index $i \geqslant 0$.

Given $\alpha = (\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5) \in \mathbb{N}^5$, let us denote $X^\alpha = X_1^{\alpha_1} X_2^{\alpha_2} X_3^{\alpha_3} X_4^{\alpha_4} X_5^{\alpha_5}$. We have that

$$\{X^\alpha : \alpha \in \mathbb{N}^5\}$$

is a $\mathbb{C}$-basis of $S$ and the subset of the former monomials with $\alpha_1+\alpha_2+\alpha_3+\alpha_4+\alpha_5 \leqslant i$ is a $\mathbb{C}$-basis of each subspace $S_i$. The number of elements of this basis is $\binom{i+5}{5}$, which is a polynomial in the indeterminate $i$ of degree five. As a consequence, the Gelfand–Kirillov dimension of $S$ is equal to five.

Similar computations, taking $\{C^{\alpha_1}(C-\lambda X_0)^{\alpha_2} X_-^{\alpha_3} X_+^{\alpha_4} : \alpha_i \in \mathbb{N}\}$ ($\{(1-\lambda X_0)^{\alpha_1} X_-^{\alpha_2} X_+^{\alpha_3} : \alpha_i \in \mathbb{N}\}$) as a $\mathbb{C}$-basis of $\mathcal{B}$ ($\mathcal{A}$), prove that $\mathrm{GKdim}(\mathcal{B}) = 4$ ($\mathrm{GKdim}(\mathcal{A}) = 3$).

### *3.4. Auslander-regular and Cohen–Macaulay conditions*

Let $R$ be a noetherian ring. An $R$-module $M$ satisfies the *Auslander condition* if for any non-negative integer $n$ and any submodule $N \subseteq \mathrm{Ext}^n_R(N,R)$ we have $j_R(N) \geqslant n$, $j_R(N)$ being the *grade* of $N$, which is defined

$$j_R(N) = \inf\{i : \mathrm{Ext}^i_R(N,R) \neq 0\} \in \mathbb{N} \cup \{\infty\}.$$

The ring $R$ satisfies the *Auslander–Gorenstein condition* (*Auslander-regular condition*) if any finitely generated $R$-module satisfies the Auslander condition and $R$ has finite left and right injective dimension (finite global dimension) (see [4]).

A noetherian $K$-algebra $R$ satisfies the *Cohen–Macaulay condition* if $\mathrm{GKdim}(R) \in \mathbb{N}$ and for every finitely generated $R$-module $M$ we have $j_R(M) + \mathrm{GKdim}(M) = \mathrm{GKdim}(R)$ (see [15]).

**Proposition 2.** *The algebra $S(\mathcal{B};\mathcal{A})$ satisfies the Auslander-regular and the Cohen–Macaulay conditions.*

**Proof.** Let us start with $S$: Auslander regularity follows directly from [17, III,3.4.6], by the construction of $S$ as an iterated Ore extension. To obtain the Cohen–Macaulay condition we graduate the base rings that constitute the chain of Ore extensions of $S$ as follows:

$$\begin{aligned}
R &:= \mathbb{C}[X_3, X_4, X_5];\\
T &:= R[X_2; \sigma_0]\\
&\quad \{R_n\}_n;\\
R_n &= \mathbb{C}\langle X_3^{\alpha_3} X_4^{\alpha_4} X_5^{\alpha_5} : \alpha_3+\alpha_4+\alpha_5 = n\rangle;\\
&\quad \sigma_0(X_3^{\alpha_3} X_4^{\alpha_4} X_5^{\alpha_5}) = q^{2\alpha_3} X_3^{\alpha_3} X_4^{\alpha_4} X_5^{\alpha_5};\\
S &= T[X_1; \sigma_1, \delta];\\
&\quad \{T_n\}_n;\\
T_n &= \mathbb{C}\langle X_2^{\alpha_2} X_3^{\alpha_3} X_4^{\alpha_4} X_5^{\alpha_5} : \alpha_2+\alpha_3+\alpha_4+\alpha_5 = n\rangle;\\
&\quad \sigma_1(X_2^{\alpha_2} X_3^{\alpha_3} X_4^{\alpha_4} X_5^{\alpha_5}) = q^{-2\alpha_3} X_2^{\alpha_2} X_3^{\alpha_3} X_4^{\alpha_4} X_5^{\alpha_5}.
\end{aligned}$$

Now applying the second part of [16, lemma] we obtain that $S$ is Cohen–Macaulay. In an analogous way, the descriptions that we obtained for $\mathcal{B}$ and $\mathcal{A}$ as iterated Ore extensions also prove the result for these two algebras. □

**Remarks.** There are alternative methods to show that $\mathcal{B}$ and $\mathcal{A}$ satisfy Auslander-regular and Cohen–Macaulay conditions.

(1) Once we have proved the Auslander-regular and Cohen–Macaulay conditions on $S$, we may also use the third part of [16, lemma] to prove these conditions on $\mathcal{B}$ and $\mathcal{A}$, because of the description of $\mathcal{B}$ as a quotient of $S$ by the central and regular element of $S$, $p$, and $\mathcal{A}$ as a quotient of $\mathcal{B}$ by the central and regular element of $\mathcal{B}$, $C-1$.

(2) Once Auslander-regular (in particular, Auslander–Gorenstein) and Cohen–Macaulay conditions are verified by an algebra $R$, we have another way of computation of injective and Gelfand–Kirillov dimensions of $R$: the existence of non-zero finite-dimensional $R$-modules and the equality $\mathrm{idim}(R) = \mathrm{GKdim}(R)$ are equivalent (see [7]).

### *3.5. Strong second-layer condition*

We first recall this technical condition and prove that $\mathcal{B}$ and $\mathcal{A}$ verify it.

Let $R$ be a noetherian ring. A prime ideal $P$ of $R$ satisfies the *left strong second-layer condition* if there is no prime ideal $Q \subsetneq P$ and an exact sequence of finitely generated uniform left $R$-modules

$$0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0,$$

such that

(i) $L = \mathrm{Ann}^r_M(P)$, where $\mathrm{Ann}^r_M(P) := \{m \in M : Pm = 0\}$, the right annihilator of $P$ in $M$,
(ii) $\mathrm{Ann}^l_R(L') = P$ for any left $R$-submodule $0 \neq L' \subseteq L$, and
(iii) $Q = \mathrm{Ann}^l_R(M) = \mathrm{Ann}^l_R(N')$ for any left $R$-submodule $0 \neq N' \subseteq N$.

The *right strong second-layer condition* is defined in a similar way. The ideal $P$ is said to have the *strong second-layer condition* if it verifies the left and right strong second-layer conditions. $R$ is said to satisfy the strong second-layer condition if all the prime ideals of $R$ verify it.

We refer mainly to [5] for relative results on the strong second-layer condition (see also [3, 8, 12, 19]).

**Proposition 3.** *$\mathcal{B}$ and $\mathcal{A}$ satisfy the strong second-layer condition.*

**Proof.** This follows directly from the description of $\mathcal{B}$ and $\mathcal{A}$ as finitely generated modules over their respective centres and Letzter's theorem (see [14, 4.2]). □

## 4. Bernstein duality

We are now going to complete the representation of the Bernstein duality by $R^0$ in $R = \mathcal{B}$ and $\mathcal{A}$ by showing that $\dim_{\mathbb{C}}(R/P) = \dim_{\mathbb{C}}(\mathrm{Ext}^{\mu}_R(R/P, R))$ for any cofinite prime ideal $P$ of $R$, where $\mu = \mathrm{GKdim}(R) = \mathrm{idim}(R)$. To do this we only need to prove that $\dim_{\mathbb{C}}(M) = \dim_{\mathbb{C}}(\mathrm{Ext}^{\mu}_R(M, R))$, for any finite-dimensional simple $R$-module $M$, because of the bijection between finite-dimensional simple $R$-modules $M$ and cofinite prime ideals of $R$ by setting the image of $M$ equal to $\mathrm{Ann}^l_R(M)$ (see [7]).

### *4.1. Irreducible finite-dimensional representations*

We recall the classification of all the finite-dimensional simple $R$-modules, for $R = \mathcal{B}$ and $\mathcal{A}$, in accordance with the results obtained in [6] and [1].

We start with $R = \mathcal{B}$. Unlike the above authors, we are using the generators of the basis (4) instead of the basis (3) to describe the action of $R$ over each one of the finite-dimensional simple $R$-modules, to make their later treatment easier.

Let $M$ be a finite-dimensional simple $\mathcal{B}$-module. Since $C$, $C'_2$, $(C-\lambda X_0)^l$, $X^l_+$, $X^l_-$ belong to the centre of $\mathcal{B}$, they act over $M$ as scalars, which we denote respectively by $c$, $c'_2$, $z$, $x^l_+$, $x^l_-$. These scalars satisfy the relation (obtained from (7))

$$x^l_- x^l_+ = \lambda^{-2l} \prod_{r=0}^{l-1} q^{-2r-1}\{-qd^2q^{2r} + (q+q^{-1})c\nu - q^{-1}\nu^2 q^{-2r}\}, \tag{8}$$

with $\nu$ a complex number such that $z = \nu^l$ and $d^2 = c^2 - \lambda^2 c'_2$.

The study of the different values of the parameters produces the complete list of the finite-dimensional simple $\mathcal{B}$-modules $M$.

*Case 1.* $z \neq 0$ and $x_- \neq 0$. $M$ is an $l$-dimensional vector space with $\mathbb{C}$-basis $\{v_0, \ldots, v_{l-1}\}$ and $\mathcal{B}$-action

$$\begin{aligned}
&C'_2 v_p = c'_2 v_p\\
&C v_p = c v_p\\
&(C - \lambda X_0) v_p = q^{2p}\nu v_p\\
&X_- v_p = x_- v_{p+1}\\
&X_+ v_p = x_-^{-1}\lambda^{-2}\{-d^2 + (1+q^{-2})c\nu q^{2p} - q^{-2}\nu^2 q^{4p}\} v_{p-1}
\end{aligned}$$

with $0 \leqslant p \leqslant l-1$ and the parameters verifying (8).

*Case 2.* $z \neq 0, x_- = 0$ and $x_+ \neq 0$. $M$ is an $l$-dimensional vector space with $\mathbb{C}$-basis $\{w_0, \ldots, w_{l-1}\}$ and $\mathcal{B}$-action

$$
\begin{aligned}
&C_2' w_p = c_2' w_p && \text{if } 0 \leqslant p \leqslant l-1;\\
&C w_p = c w_p && \text{if } 0 \leqslant p \leqslant l-1;\\
&(C - \lambda X_0) w_p = q^{-2p} \nu w_p && \text{if } 0 \leqslant p \leqslant l-1;\\
&X_+ w_p = x_+ w_{p+1} && \text{if } 0 \leqslant p \leqslant l-1;\\
&X_- w_0 = 0;\\
&X_- w_p = x_+^{-1} \lambda^{-2} \{-d^2 + (1+q^2) c \nu q^{-2p} - q^2 \nu^2 q^{-4p}\} w_{p-1}\\
& && \text{if } 1 \leqslant p \leqslant l-1,
\end{aligned}
$$

with $c_2' = \lambda^{-2}\{c^2 - (1+q^2)c\nu + q^2\nu^2\}$.

*Case 3.* $z \neq 0$ and $x_- = x_+ = 0$. (Highest-weight and lowest-weight representations.)
In this case, we have two types of finite-dimensional simple $\mathcal{B}$-modules.

(1) $M$ is an $l$-dimensional vector space with $\mathbb{C}$-basis $\{v_0, \ldots, v_{l-1}\}$ and $\mathcal{B}$-action

$$
\begin{aligned}
&C_2' v_p = c_2' v_p && \text{if } 0 \leqslant p \leqslant l-1;\\
&C v_p = c v_p && \text{if } 0 \leqslant p \leqslant l-1;\\
&(C - \lambda X_0) v_p = q^{2p} \nu v_p && \text{if } 0 \leqslant p \leqslant l-1;\\
&X_- v_p = v_{p+1} && \text{if } 0 \leqslant p \leqslant l-2;\\
&X_- v_{l-1} = 0;\\
&X_+ v_0 = 0;\\
&X_+ v_p = \lambda^{-1}[p] q^{p-2} \nu((1+q^2)c - (1+q^{2p})\nu) v_{p-1} && \text{if } 1 \leqslant p \leqslant l-1,
\end{aligned}
$$

with $\nu \neq 0$ and $c$ such that $(1+q^2)c \neq (1+q^{2p})\nu$ for all $p \in \{1, ..., l-1\}$. Moreover $c_2' = \lambda^{-2}\{c^2 - (1+q^{-2})c\nu + q^{-2}\nu^2\}$.

(2) For each $n \in \{1, \ldots, l-1\}$, $M$ is a $n$-dimensional vector space with $\mathbb{C}$-basis $\{v_0, \ldots, v_{n-1}\}$ and $\mathcal{B}$-action

$$
\begin{aligned}
&C_2' v_p = c_2' v_p && \text{if } 0 \leqslant p \leqslant n-1;\\
&C v_p = c v_p && \text{if } 0 \leqslant p \leqslant n-1;\\
&(C - \lambda X_0) v_p = q^{2p} \nu v_p && \text{if } 0 \leqslant p \leqslant n-1;\\
&X_- v_p = v_{p+1} && \text{if } 0 \leqslant p \leqslant n-2;\\
&X_- v_{n-1} = 0;\\
&X_+ v_0 = 0;\\
&X_+ v_p = \lambda^{-1}[p] q^{p-2} \nu((1+q^2)c - (1+q^{2p})\nu) v_{p-1} && \text{if } 1 \leqslant p \leqslant n-1,
\end{aligned}
$$

with the restriction $(1+q^2)c = (1+q^{2n})\nu$ and $\nu \neq 0$. Moreover $c_2' = \lambda^{-2}\{c^2 - (1+q^{-2})c\nu + q^{-2}\nu^2\}$.

*Case 4.* $z = 0$. $M$ is a unidimensional vector space with $\mathcal{B}$-action over the basis $\{v\}$

$$
\begin{aligned}
&C_2' v = c_2' v\\
&C v = c v\\
&(C - \lambda X_0) v = 0\\
&X_- v = x_- v\\
&X_+ v_p = x_+ v
\end{aligned}
$$

with $x_+ x_- = c_2' - \lambda^{-2} c^2 = -\lambda^{-2} d^2$.

This completes the list of all the finite-dimensional simple $\mathcal{B}$-modules. The corresponding one for $\mathcal{A}$ is obtained from the previous classification by taking $c = 1$ in all the cases where this is possible, because of the isomorphism

$$\mathcal{A} \cong \mathcal{B}/(C-1).$$

### 4.2. Dimension of Ext

Using some computational techniques of non-commutative Groebner bases as the first author developed in [9], we are going to calculate the $\mathbb{C}$-dimension of $\mathrm{Ext}^{\mu}_{R}(M, R)$, for each finite-dimensional simple $R$-module $M$, where $\mu = \mathrm{GKdim}(R) = \mathrm{idim}(R)$. We shall prove that this dimension coincides, in all cases, with $\dim_{\mathbb{C}}(M)$.

Start with $R = \mathcal{B}$. Let $M$ be a finite-dimensional simple $\mathcal{B}$-module. We have to prove that

$$\dim_{\mathbb{C}}(\mathrm{Ext}^4_{\mathcal{B}}(M, \mathcal{B})) = \dim_{\mathbb{C}}(M).$$

Since $\mathcal{B} \cong S/Sp$, we may consider $M$ as an $S$-module by the corresponding change ring. As $p$ is a central and regular element of $S$, by Rees's theorem (see [20, 9.37]) we have

$$\mathrm{Ext}^4_{\mathcal{B}}(M, \mathcal{B}) \cong \mathrm{Ext}^5_S(M, S).$$

Then, our problem is to compute the $\mathbb{C}$-dimension of $\mathrm{Ext}^5_S(M, S)$. Let us study it for each of the types of finite-dimensional simple $\mathcal{B}$-module obtained above.

*Case 1.* $z \neq 0$ and $x_- \neq 0$. In this case, the $l$-dimensional $\mathcal{B}$-module $M$ is isomorphic, as a $S$-module, to

$$M \cong \frac{S}{S(X_1^l - x_+^l, X_2^l - x_-^l, X_3 - \nu, X_4 - c, X_5 - c_2', p)}$$

where $c, c_2', \nu, x_+^l, x_-^l \in \mathbb{C}$ and are related by (8).

To compute $\mathrm{Ext}^5_S(M, S)$ we consider the Groebner basis theory on the ring $S$ as it appears in [9]. The left $S$-ideal $Q = S(X_1^l - x_+^l, X_2^l - x_-^l, X_3 - \nu, X_4 - c, X_5 - c_2', p)$ is also described by $Q = S(X_1^l - x_+^l, X_2^l - x_-^l, X_3 - \nu, X_4 - c, X_5 - c_2', X_1 - x_-^{-l}\beta X_2^{l-1})$, with $\beta = c_2' - \lambda^{-2}(c^2 - (1+q^{-2})\nu c + q^{-2}\nu^2)$. Let us call $G_1 = X_1 - x_-^{-l}\beta X_2^{l-1}$, $G = X_1^l - x_+^l$, $G_2 = X_2^l - x_-^l$, $G_3 = X_3 - \nu$, $G_4 = X_4 - c$, $G_5 = X_5 - c_2'$. We can check that

$$\mathbb{G} = \{G_1, G_2, G_3, G_4, G_5\},$$

is a Groebner basis of $Q$.

Let us consider a free presentation $\varphi_1 : S^5 \to Q$ of $Q$ by setting $\varphi_1(e_i) = G_i$, $(i = 1, \ldots, 5)$.

In order to compute $\mathrm{Ker}(\varphi_1)$ we write the semisyzygies $S(G_i, G_j)$, $i < j$, and divide them with respect to the Groebner basis $\mathbb{G}$:

$$S(G_1, G_2) = X_2^l G_1 - X_1 G_2 = x_-^l G_1 - x_-^{-l}\beta X_2^{l-1} G_2$$
$$S(G_1, G_3) = q^2 X_3 G_1 - X_1 G_3 = \nu G_1 - x_-^{-l}\beta X_2^{l-1} G_3$$
$$S(G_1, G_4) = X_4 G_1 - X_1 G_4 = c G_1 - x_-^{-l}\beta X_2^{l-1} G_4$$
$$S(G_1, G_5) = X_5 G_1 - X_1 G_5 = c_2' G_1 - x_-^{-l}\beta X_2^{l-1} G_5$$
$$S(G_2, G_3) = X_3 G_2 - X_2^l G_3 = \nu G_2 - x_-^l G_3$$
$$S(G_2, G_4) = X_4 G_2 - X_2^l G_4 = c G_2 - x_-^l G_4$$
$$S(G_2, G_5) = X_5 G_2 - X_2^l G_5 = c_2' G_2 - x_-^l G_5$$
$$S(G_3, G_4) = X_4 G_3 - X_3 G_4 = c G_3 - \nu G_4$$

$$S(G_3, G_5) = X_5G_3 - X_3G_5 = c_2'G_3 - \nu G_5$$
$$S(G_4, G_5) = X_5G_4 - X_4G_5 = c_2'G_4 - cG_5.$$

If $S(G_i, G_j) = C_{ij}G_i - C_{ji}G_j = \sum_h Q_{ijh}G_h$, where $C_{ij}, C_{ji}$ are coefficients in $S$ and define $s_{ij} = C_{ij}e_i - C_{ji}e_j - \sum_h Q_{ijh}e_h$, then it is well known that the set $\{s_{ij} : 1 \leqslant i < j \leqslant 5\}$ is a system of generators of $\mathrm{Ker}(\varphi_1)$; in fact it is a Groebner basis with respect to a monomial order (see [9]).

Let us define

$$H_1 = s_{12} = (X_2^l - x_-^l, -X_1 + x_-^{-l}\beta X_2^{l-1}, 0, 0, 0)$$
$$H_2 = s_{13} = (q^2X_3 - \nu, 0, -X_1 + x_-^{-l}\beta X_2^{l-1}, 0, 0)$$
$$H_3 = s_{14} = (X_4 - c, 0, 0, -X_1 + x_-^{-l}\beta X_2^{l-1}, 0)$$
$$H_4 = s_{15} = (X_5 - c_2', 0, 0, 0, -X_1 + x_-^{-l}\beta X_2^{l-1})$$
$$H_5 = s_{23} = (0, X_3 - \nu, -X_2^l + x_-^l, 0, 0)$$
$$H_6 = s_{24} = (0, X_4 - c, 0, -X_2^l + x_-^l, 0)$$
$$H_7 = s_{25} = (0, X_5 - c_2', 0, 0, -X_2^l + x_-^l)$$
$$H_8 = s_{34} = (0, 0, X_4 - c, -X_3 + \nu, 0)$$
$$H_9 = s_{35} = (0, 0, X_5 - c_2', 0, -X_3 + \nu)$$
$$H_{10} = s_{45} = (0, 0, 0, X_5 - c_2', -X_4 + c).$$

With this notation $\mathbb{H} = \{H_1, \ldots, H_{10}\}$ is a Groebner basis of $\mathrm{Ker}(\varphi_1)$. We consider now a free presentation $\varphi_2 : S^{10} \longrightarrow \mathrm{Ker}(\varphi_1)$ of $\mathrm{Ker}(\varphi_1)$ by setting $\varphi_2(e_i) = H_i$ ($i = 1, \ldots, 10$). In order to compute $\mathrm{Ker}(\varphi_2)$ we consider the minimum common multiple $X_{ij}$ of $H_i$ and $H_j$, $i < j$. The only non-zero ones are

$$X_{12} = X_2^lX_3 \qquad X_{13} = X_2^lX_4 \qquad X_{14} = X_2^lX_5 \qquad X_{23} = X_3X_4$$
$$X_{24} = X_3X_5 \qquad X_{34} = X_4X_5 \qquad X_{56} = X_3X_4 \qquad X_{57} = X_3X_5$$
$$X_{67} = X_4X_5 \qquad X_{89} = X_4X_5.$$

As a consequence $\mathrm{Ker}(\varphi_2)$ has ten generators, which can be described as

$$I_1 = (X_3 - q^{-2}\nu, q^{-2}(-X_2^l + x_-^l), 0, 0, q^{-2}(X_1 - x_-^{-l}\beta X_2^{l-1}), 0, 0, 0, 0, 0)$$
$$I_2 = (X_4 - c, 0, -X_2^l + x_-^l, 0, 0, X_1 - x_-^{-l}\beta X_2^{l-1}, 0, 0, 0, 0)$$
$$I_3 = (X_5 - c_2', 0, 0, -X_2^l + x_-^l, 0, 0, X_1 - x_-^{-l}\beta X_2^{l-1}, 0, 0, 0)$$
$$I_4 = (0, q^{-2}(X_4 - c), -X_3 + \nu q^{-2}, 0, 0, 0, 0, q^{-2}(X_1 - x_-^{-l}\beta X_2^{l-1}), 0, 0)$$
$$I_5 = (0, q^{-2}(X_5 - c_2'), 0, -X_3 + \nu q^{-2}, 0, 0, 0, 0, q^{-2}(X_1 - x_-^{-l}\beta X_2^{l-1}), 0)$$
$$I_6 = (0, 0, X_5 - c_2', -X_4 + c, 0, 0, 0, 0, 0, X_1 - x_-^{-l}\beta X_2^{l-1})$$
$$I_7 = (0, 0, 0, 0, X_4 - c, -X_3 + \nu, 0, X_2^l - x_-^l, 0, 0)$$
$$I_8 = (0, 0, 0, 0, X_5 - c_2', 0, -X_3 + \nu, 0, X_2^l - x_-^l, 0)$$
$$I_9 = (0, 0, 0, 0, 0, X_5 - c_2', -X_4 + c, 0, 0, X_2^l - x_-^l)$$
$$I_{10} = (0, 0, 0, 0, 0, 0, 0, X_5 - c_2', -X_4 + c, X_3 - \nu).$$

Again, we have that $\mathbb{I} = \{I_1, \ldots, I_{10}\}$ is a Groebner basis of $\mathrm{Ker}(\varphi_2)$ with respect to a particular monomial order. We continue the process and define a free presentation $\varphi_3 : S^{10} \longrightarrow \mathrm{Ker}(\varphi_2)$ of $\mathrm{Ker}(\varphi_2)$ by setting $\varphi_3(e_i) = I_i$ ($i = 1, \ldots, 10$). Computing the minimum common multiples $X_{ij}$ of $I_i$ and $I_j$, $i < j$, we have that the only non-zero ones are

$$X_{12} = X_3X_4 \qquad X_{13} = X_3X_5 \qquad X_{23} = X_4X_5$$
$$X_{45} = X_4X_5 \qquad X_{78} = X_4X_5.$$

Thus $\mathrm{Ker}(\varphi_3)$ has five generators, which are

$$J_1 = (X_4 - c, -X_3 + q^{-2}\nu, 0, X_2^l - x_-^l, 0, 0, q^{-2}(-X_1 + x_-^{-l}\beta X_2^{l-1}), 0, 0, 0)$$
$$J_2 = (X_5 - c_2', 0, -X_3 + q^{-2}\nu, 0, X_2^l - x_-^l, 0, 0, q^{-2}(-X_1 + x_-^{-l}\beta X_2^{l-1}), 0, 0)$$
$$J_3 = (0, X_5 - c_2', -X_4 + c, 0, 0, X_2^l - x_-^l, 0, 0, -X_1 + x_-^{-l}\beta X_2^{l-1}, 0)$$
$$J_4 = (0, 0, 0, q^2(X_5 - c_2'), q^2(-X_4 + c), q^2X_3 - \nu, 0, 0, 0, -X_1 + x_-^{-l}\beta X_2^{l-1})$$
$$J_5 = (0, 0, 0, 0, 0, 0, X_5 - c_2', -X_4 + c, X_3 - \nu, -X_2^l + x_-^l).$$

Again, $\mathbb{J} = \{J_1, \ldots, J_5\}$ is a Groebner basis of $\mathrm{Ker}(\varphi_3)$ with respect to a particular monomial order. We now consider a free presentation $\varphi_4 : S^5 \longrightarrow \mathrm{Ker}(\varphi_3)$ of $\mathrm{Ker}(\varphi_3)$ by setting $\varphi_4(e_i) = J_i$ $(i = 1, \ldots, 5)$. If we compute the minimum common multiple $X_{ij}$ of $J_i$ and $J_j$, $i < j$, the only non-zero one is $X_{12} = X_4X_5$. Thus $\mathrm{Ker}(\varphi_4)$ has one generator, that is

$$L = (X_5 - c_2', -X_4 + c, X_3 - q^{-2}\nu, q^{-2}(-X_2^l + x_-^l), q^{-2}(X_1 - x_-^{-l}\beta X_2^{l-1}))$$

and $\mathbb{L} = \{L\}$ is a Groebner basis of $\mathrm{Ker}(\varphi_4)$.

Putting together all this information, we can build a free resolution of $S/Q$ as follows:

$$\begin{array}{ccccccccccccc}
\mathrm{Ker}(\varphi_4) & \longrightarrow & S^5 & \longrightarrow & S^{10} & \longrightarrow & S^{10} & \longrightarrow & S^5 & \longrightarrow & S & \longrightarrow & S/Q \\
 & & & \searrow^{\varphi_4} & \uparrow & \searrow^{\varphi_3} & \uparrow & \searrow^{\varphi_2} & \uparrow & \searrow^{\varphi_1} & \uparrow & & \\
 & & & & \mathrm{Ker}(\varphi_3) & & \mathrm{Ker}(\varphi_2) & & \mathrm{Ker}(\varphi_1) & & Q & &
\end{array}$$

Hence $\mathrm{Ext}_S^5(S/Q, S) \cong \mathrm{Ext}_S^1(\mathrm{Ker}(\varphi_3), S)$. Taking the free presentation of $\mathrm{Ker}(\varphi_3)$

$$0 \longrightarrow \mathrm{Ker}(\varphi_4) \stackrel{\phi}{\longrightarrow} S^5 \stackrel{\varphi_4}{\longrightarrow} \mathrm{Ker}(\varphi_3) \longrightarrow 0$$

we have a long exact sequence

$$\cdots \longrightarrow \mathrm{Hom}_S(S^5, S) \stackrel{\phi^*}{\longrightarrow} \mathrm{Hom}_S(\mathrm{Ker}(\varphi_4), S) \longrightarrow \mathrm{Ext}_S^1(\mathrm{Ker}(\varphi_3), S) \longrightarrow 0 \ldots.$$

Then

$$\mathrm{Ext}_S^1(\mathrm{Ker}(\varphi_3), S) \cong \frac{\mathrm{Hom}_S(\mathrm{Ker}(\varphi_4), S)}{\mathrm{Im}(\phi^*)}$$
$$\cong \frac{S}{(X_5 - c_2', -X_4 + c, X_3 - q^{-2}\nu, q^{-2}(-X_2^l + x_-^l), q^{-2}(X_1 - x_-^{-l}\beta X_2^{l-1}))S}.$$

As a consequence, the $\mathbb{C}$-dimension of $\mathrm{Ext}_S^5(M, S)$ is $l$.

*Case 2.* $z \neq 0, x_- = 0, x_+ \neq 0$. In this case, the $l$-dimensional $\mathcal{B}$-module $M$ is isomorphic, as a $S$-module, to the quotient

$$M \cong \frac{S}{S(X_1^l - x_+^l, X_2, X_3 - \nu, X_4 - c, X_5 - c_2')}$$

with $c_2', c, \nu, x_+^l \in \mathbb{C}$ related by $c_2' = \lambda^{-2}\{c^2 - (1 + q^2)c\nu + q^2\nu^2\}$. After developing similar calculations in the previous case we obtain an isomorphism

$$\mathrm{Ext}_S^5(M, S) \cong \frac{S}{(q^{-2}(X_5 - c_2'), q^{-2}(-X_4 + c), q^{-2}(X_3 - \nu q^2), -X_2, X_1^l - x_+^l)S}$$

from which

$$\dim_{\mathbb{C}}(\mathrm{Ext}_S^5(M, S)) = l = \dim_{\mathbb{C}}(M).$$

*Case 3.* $z \neq 0, x_- = x_+ = 0$. In this case we obtained two types of finite-dimensional simple $\mathcal{B}$-module.

(1) In this case, the $l$-dimensional $\mathcal{B}$-module $M$ is isomorphic, as a $S$-module, to

$$M \cong \frac{S}{S(X_1, X_2^l, X_3 - \nu, X_4 - c, X_5 - c_2')}$$

where all the parameters are complex numbers and verify that $\nu \neq 0$, $(1 + q^2)c \neq (1 + q^{2p})\nu$ for all $p \in \{1, \dots, l-1\}$ and $c_2' = \lambda^{-2}\{c^2 - (1 + q^{-2})c\nu + q^{-2}\nu^2\}$.

Operating similarly to the above cases we obtain an isomorphism

$$\mathrm{Ext}_S^5(M, S) \cong \frac{S}{(X_5 - c_2', -X_4 + c, X_3 - q^{-2}\nu, -q^{-2}X_2^l, q^{-2}X_1)S}.$$

Hence

$$\dim_{\mathbb{C}}(\mathrm{Ext}_S^5(M, S)) = l = \dim_{\mathbb{C}}(M).$$

(2) In this case, the $n$-dimensional $\mathcal{B}$-module that we obtained for each $n \in \{1, \dots, l-1\}$ is isomorphic, as a $S$-module, to

$$M \cong \frac{S}{S(X_1, X_2^n, X_3 - \nu, X_4 - c, X_5 - c_2')}$$

where all the parameters are complex numbers and verify $(1 + q^2)c = (1 + q^{2n})\nu$, $\nu \neq 0$ and $c_2' = \lambda^{-2}\{c^2 - (1 + q^{-2})c\nu + q^{-2}\nu^2\}$.

Again, operating in a similar way to the former cases, we obtain an isomorphism

$$\mathrm{Ext}_S^5(M, S) \cong \frac{S}{(X_5 - c_2', -X_4 + c, X_3 - q^{-2}q^{2n}\nu, -q^{2n}X_2^n, q^{-2}X_1)S}$$

so

$$\dim_{\mathbb{C}}(\mathrm{Ext}_S^5(M, S)) = n = \dim_{\mathbb{C}}(M).$$

*Case 4.* $z = 0$. In this case, the $l$-dimensional $\mathcal{B}$-module $M$ is isomorphic, as a $S$-module, to

$$M \cong \frac{S}{S(X_1 - x_+, X_2 - x_-, X_3, X_4 - c, X_5 - c_2')}$$

where all the parameters are complex numbers and verify $x_+x_- = c_2' - \lambda^{-2}c^2 = -\lambda^{-2}d^2$.

After developing similar computations as in the former cases we obtain an isomorphism

$$\mathrm{Ext}_S^5(M, S) \cong \frac{S}{(X_5 - c_2', -X_4 + c, X_3, -q^2X_2 + x_-, q^{-2}X_1 - x_+)S}$$

from which

$$\dim_{\mathbb{C}}(\mathrm{Ext}_S^5(M, S)) = 1 = \dim_{\mathbb{C}}(M).$$

As a consequence of all the above results we have proved the following theorem.

**Theorem 4.** *Let $M$ be a finite-dimensional simple $\mathcal{B}$-module. Then*

$$\dim_{\mathbb{C}}(M) = \dim_{\mathbb{C}}(\mathrm{Ext}_{\mathcal{B}}^4(M, \mathcal{B})).$$

Furthermore, taking into account the isomorphism $\mathcal{A} \cong \mathcal{B}/(C - 1)$ and Rees's theorem again:

**Corollary 5.** *Let $M$ be a finite-dimensional simple $\mathcal{A}$-module. Then*

$$\dim_{\mathbb{C}}(M) = \dim_{\mathbb{C}}(\mathrm{Ext}_{\mathcal{A}}^3(M, \mathcal{A})).$$

As a consequence, since for each cofinite prime ideal $P$ of $R$ it is verified that $R/P \cong M^m$, for some finite-dimensional simple $R$-module $M$ and $m \in \mathbb{N}$, we have the desired result.

**Corollary 6.** *Let $P$ be a cofinite prime ideal of either $R = \mathcal{B}$ or $\mathcal{A}$. Then*

$$\dim_{\mathbb{C}}(R/P) = \dim_{\mathbb{C}}(\mathrm{Ext}^{\mu}_{R}(R/P, R))$$

*where* $\mu = \mathrm{GKdim}(R) = \mathrm{idim}(R)$.

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